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Positive energy unitary irreducible representations of $D = 6$ conformal supersymmetry

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Abstract

We give a constructive classification of the positive energy (lowest weight) unitary irreducible representations of the $D = 6$ superconformal algebras $osp(8^*/2N)$. Our results confirm all but one of the conjectures of Minwalla (for $N = 1, 2$) on this classification. Our main tool is the explicit construction of the norms of the states that have to be checked for positivity. We also give the reduction of the four exceptional unitary irreducible representations.

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1. Introduction

Recently, superconformal field theories in various dimensions have been attracting increased interest, in particular, due to their duality to AdS supergravities, cf [1–35] and references therein. Particularly important are those for $D \leqslant 6$ since in these cases the relevant superconformal algebras satisfy [36] the Haag–Lopuszanski–Sohnius theorem [37]. This makes the classification of the unitary irreducible representations (UIRs) of these superalgebras very important. Until recently such classification was known only for the $D = 4$ superconformal algebras $su(2, 2/N)$ [38] (for $N = 1$) [39–42]. Recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [43], but some of the results were conjectural and there was not enough detail in order to check these conjectures. On the other hand the applications of $D = 6$ unitary irreps require firmer theoretical basis. Among the many interesting applications we shall mention the analysis of OPEs and $1/2$ BPS operators [18, 29, 35]. In particular, it is important that some general properties of abstract superconformal field theories can be obtained by using the BPS nature of a certain class of superconformal primary operators and the model-independent nature of superconformal OPEs. In the classification of UIRs of superconformal algebras an important role is played by the representations with 'quantized' conformal dimension since in the quantum field theory framework they correspond to operators with 'protected' scaling dimension and therefore imply 'non-renormalization theorems' at the quantum level.

Motivated by the above we decided to re-examine the list of UIRs of the $D = 6$ superconformal algebras in detail. More than that, we treat the superalgebras $osp(8^*/2N)$ for arbitrary N. Thus, we give the final list of UIRs for $D = 6$. With this we also confirm all but one of the conjectures of [43] for $N = 1, 2$. Our main tool is the explicit construction of the norms. This, on the one hand, enables us to prove the unitarity list and, on the other, enables us to give the states of the irreps explicitly.

The paper is organized as follows. In section 2 we discuss in detail the lowest weight representations of the superalgebras $osp(8^*/2N)$. In particular, we define explicitly the norm squared of the states that has to be checked for positivity. In section 3 we state the main result (theorem) on the lowest weight (positive energy) UIRs and show explicitly the proof of necessity. (After the theorem we comment exactly on the results of [43] giving also the relation between our notations.) We also give the general form of the norms which is enough for the proof of sufficiency. For part of the states (the fully factorizable ones) we give the norms explicitly, for the rest (the unfactorizable ones) the formulae are very involved and in general only recursive. These results are in the generic situation. In section 4 we show the unitarity at the four exceptional points. We give explicitly the states of zero norm (though not all for $N > 1$), which have to be decoupled for the unitary irrep. In section 5 we discuss the ongoing research.

2. Representations of D = 6 conformal supersymmetry

2.1. The setting

Our basic reference for Lie superalgebras is [44]. The superconformal algebras in $D = 6$ are $\mathcal{G} = osp(8^*/2N)$. We label their physically relevant representations by the signature:

$$\chi = [d; n_1, n_2, n_3; a_1, \ldots, a_N] \tag{2.1}$$

where d is the conformal weight, n_1, n_2, n_3 are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the $D = 6$ Lorentz algebra $so(5, 1)$ and $a_1, \ldots, a_N$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or R) symmetry algebra $usp(2N)$. The even subalgebra of $osp(8^*/2N)$ is the algebra $so^*(8) \oplus usp(2N)$, and $so^*(8) \cong so(6, 2)$ is the $D = 6$ conformal algebra.

Our aim is to give a constructive proof for the UIRs of $osp(8^*/2N)$ following the methods used for the $D = 4$ superconformal algebras $su(2, 2/N)$, cf [40–42]. The main tool is an adaptation of the Shapovalov form on the Verma modules V^χ over the complexification $\mathcal{G}^{\mathbb{C}} = osp(8/2N)$ of $\mathcal{G}$.

2.2. Verma modules

To introduce Verma modules we use the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{2.2}$$

where $\mathcal{G}^+, \mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that:

$$\begin{aligned} &Z v_0 = 0 \qquad && Z \in \mathcal{G}^- \\ &H v_0 = \Lambda(H) v_0 \qquad && H \in \mathcal{H} \end{aligned} \tag{2.3}$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

The lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathcal{H}$. In order to have Λ corresponding to χ, one can choose a basis in $\mathcal{H}$ so as to obtain the entries in the signature χ by evaluating Λ on the basis elements of $\mathcal{H}$.

2.3. Root systems

In order to explain how the above is done we recall some facts about $osp(8/2N)$ (denoted $D(4, N)$ in [44])[1]. Their root systems are given in terms of $\epsilon_1 \ldots, \epsilon_4, \delta_1 \ldots, \delta_N$, $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta_{\hat\imath}, \delta_{\hat\jmath}) = -\delta_{\hat\imath\hat\jmath}$, $(\epsilon_i, \delta_{\hat\jmath}) = 0$. The indices $i, j, \ldots$ will take values in the set $\{1, 2, 3, 4\}$, the indices $\hat\imath, \hat\jmath, \ldots$ will take values in the set $\{1, \ldots, N\}$. The even and odd root systems are [44]:

$$\Delta_{\bar 0} = \{\pm\epsilon_i \pm \epsilon_j, i < j, \pm\delta_{\hat\imath} \pm \delta_{\hat\jmath}, \hat\imath < \hat\jmath, \pm 2\delta_{\hat\imath}\} \qquad \Delta_{\bar 1} = \{\pm\epsilon_i \pm \delta_{\hat\jmath}\} \tag{2.4}$$

(we recall that the signs $\pm$ are not correlated)[2]. We shall use the following simple root system [44]:

$$\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \delta_1, \delta_1 - \delta_2, , \ldots, \delta_{N-1} - \delta_N, 2\delta_N\} \tag{2.5a}$$

or introducing standard notation for the simple roots:

$$\begin{aligned}
&\Pi = \{\alpha_1, \ldots, \alpha_{4+n}\} \\
&\alpha_j = \epsilon_j - \epsilon_{j+1} \qquad j = 1, 2, 3 \\
&\alpha_4 = \epsilon_4 - \delta_1 \\
&\alpha_{4+\hat\jmath} = \delta_{\hat\jmath} - \delta_{\hat\jmath+1} \qquad \hat\jmath = 1, \ldots, N-1 \\
&\alpha_{4+N} = 2\delta_N
\end{aligned} \tag{2.5b}$$

The root $\alpha_4 = \epsilon_4 - \delta_1$ is odd, the other simple roots are even. For future use we also need the positive root system corresponding to Π:

$$\Delta^+_{\bar 0} = \{\epsilon_i \pm \epsilon_j, i < j, \delta_{\hat\imath} \pm \delta_{\hat\jmath}, \hat\imath < \hat\jmath, 2\delta_{\hat\imath}\} \qquad \Delta^+_{\bar 1} = \{\epsilon_i \pm \delta_{\hat\jmath}\} \tag{2.6}$$

2.4. Basis of the Cartan subalgebra

Let us denote by H_A the generators of the Cartan subalgebra, $A = 1, \ldots, 4 + N$. There is a standard choice for these generators [44]. Namely, for every even simple root α_A we choose a generator H_A so that the following equality is valid for arbitrary $\mu \in \mathcal{H}^*$:

$$\mu(H_A) = \left(\mu, \alpha_A^\vee\right) \qquad A \neq 4 \tag{2.7}$$

where $\alpha_A^\vee \equiv 2\alpha_A/(\alpha_A, \alpha_A)$. Because these H_A correspond to the simple even roots, which define the Dynkin labelling, we have the following relation with the signature χ:

$$\Lambda(H_A) = \begin{cases} -n_A & A = 1, 2, 3 \\ -a_{A-4} & A = 5, \ldots, N+4. \end{cases} \tag{2.8}$$

The minus signs are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [44]) and to Verma module reducibility w.r.t. the roots α_A (this is explained in detail in [41]).

We have not fixed only the generator H_4. The standard choice [44] is a generator corresponding to the odd simple root α_4, but we can take any element of the Cartan subalgebra which is not a linear combination of the established already $N + 3$ generators H_A. Our choice

[1] These initial facts can be given for $osp(2M/2N) = D(M, N)$ in a very similar fashion.

[2] The roots $\pm\epsilon_i \pm \epsilon_j$ provide the root system of $so(8;\mathbb{C})$, the roots $\pm\delta_i \pm \delta_j$ and $\pm 2\delta_i$ provide the root system of $sp(2N;\mathbb{C})$.

is to take the generator H_4 which corresponds to the root $\epsilon_3 + \epsilon_4$ and which together with $\alpha_1, \alpha_2, \alpha_3$ provides the root system of $so(8;\mathbb{C})$.[3] The value $\Lambda(H_4)$ cannot be a non-positive integer like the other $\Lambda(H_A)$ given in (2.8), since then we would obtain finite-dimensional representations of $so(8,\mathbb{C})$ [45], and thus, non-unitary representations of $so(6,2)$. In fact, unitarity w.r.t. $so(6,2)$ would already require that $\Lambda(H_4)$ is a non-negative number related to the physically relevant conformal weight d, which is related to the eigenvalue of the conformal Hamiltonian. That is why the lowest weight UIRs are also called *positive energy* UIRs. Here we omit the analysis by which it turns out that $\Lambda(H_4)$ differs from d by the quantity $(n_1 + 2n_2 + n_3)/2$ (which is the value of the conformal Hamiltonian of the algebra $so(5,1)$ mentioned above). Thus, we set

$$\Lambda(H_4) = d + \tfrac{1}{2}(n_1 + 2n_2 + n_3) = (\Lambda, (\epsilon_3 + \epsilon_4)^\vee) = (\Lambda, \epsilon_3 + \epsilon_4). \tag{2.9}$$

This choice is consistent with that in [43], and the usage in [18].

Having in hand the values of Λ on the basis we can recover them for any element of $\mathcal{H}$ and $\mathcal{H}^*$. In particular, for the values on the elementary functionals we have from (2.8) and (2.9):

$$\begin{aligned}
(\Lambda, \epsilon_1) &= \tfrac{1}{2}d - \tfrac{1}{4}(3n_1 + 2n_2 + n_3)\\
(\Lambda, \epsilon_2) &= \tfrac{1}{2}d + \tfrac{1}{4}(n_1 - 2n_2 - n_3)\\
(\Lambda, \epsilon_3) &= \tfrac{1}{2}d + \tfrac{1}{4}(n_1 + 2n_2 - n_3)\\
(\Lambda, \epsilon_4) &= \tfrac{1}{2}d + \tfrac{1}{4}(n_1 + 2n_2 + 3n_3)\\
(\Lambda, \delta_{\hat{j}}) &= a_{\hat{j}} + a_{\hat{j}+1} + \cdots + \alpha_N \equiv r_{\hat{j}}
\end{aligned} \tag{2.10}$$

Using (2.8 and (2.9) one can easily write $\Lambda = \Lambda(\chi)$ as a linear combination of the simple roots or of the elementary functionals $\epsilon_j, \delta_{\hat{j}}$, but this is not necessary in what follows.

2.5. Reducibility of Verma modules

Having established the relation between χ and Λ we turn our attention to the question of unitarity. The conditions of unitarity are intimately related with the conditions for reducibility of the Verma modules w.r.t. to the odd positive roots. A Verma module V^Λ is reducible w.r.t. the odd positive root γ iff the following holds [44]:

$$(\Lambda - \rho, \gamma) = 0 \qquad \gamma \in \Delta^+_{\bar{1}} \tag{2.11}$$

where $\rho \in \mathcal{H}^*$ is the very important in representation theory element given by the difference of the half-sums $\rho_{\bar{0}}, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf (2.6)):

$$\begin{aligned}
\rho &\doteq \rho_{\bar{0}} - \rho_{\bar{1}}\\
\rho_{\bar{0}} &= 3\epsilon_1 + 2\epsilon_2 + \epsilon_3 + N\delta_1 + (N-1)\delta_2 + \cdots + 2\delta_{N-1} + \delta_N\\
\rho_{\bar{1}} &= N(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4).
\end{aligned} \tag{2.12}$$

To make (2.11) explicit we need the values of Λ and ρ on the positive odd roots $\epsilon_i \pm \delta_j$ (which we obtain from (2.10)):

$$(\Lambda, \epsilon_1 \pm \delta_{\hat{j}}) = \tfrac{1}{2}d - \tfrac{1}{4}(3n_1 + 2n_2 + n_3) \pm r_{\hat{j}} \tag{2.13a}$$

$$(\Lambda, \epsilon_2 \pm \delta_{\hat{j}}) = \tfrac{1}{2}d + \tfrac{1}{4}(n_1 - 2n_2 - n_3) \pm r_{\hat{j}} \tag{2.13b}$$

$$(\Lambda, \epsilon_3 \pm \delta_{\hat{j}}) = \tfrac{1}{2}d + \tfrac{1}{4}(n_1 + 2n_2 - n_3) \pm r_{\hat{j}} \tag{2.13c}$$

$$(\Lambda, \epsilon_4 \pm \delta_{\hat{j}}) = \tfrac{1}{2}d + \tfrac{1}{4}(n_1 + 2n_2 + 3n_3) \pm r_{\hat{j}} \tag{2.13d}$$

$$(\rho, \epsilon_i \pm \delta_{\hat{j}}) = 4 - i - N \mp (N - \hat{j} + 1). \tag{2.14}$$

[3] However, in the $osp(8/2N)$ root system we have: $\epsilon_3 + \epsilon_4 = \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{N+3} + \alpha_{N+4}$.

Consequently we find that the Verma module $V^{\Lambda(\chi)}$ is reducible if the conformal weight takes one of the following $8N$ values $d^{\pm}_{i\hat{j}}$ labelled by the respective odd root $\epsilon_i \pm \delta_{\hat{j}}$:

$$d = d^{\pm}_{1\hat{j}} \doteq \tfrac{1}{2}(3n_1 + 2n_2 + n_3) + 2(3 - N) \mp 2(r_{\hat{j}} + N - \hat{j} + 1) \tag{2.15a}$$

$$d = d^{\pm}_{2\hat{j}} \doteq \tfrac{1}{2}(n_3 + 2n_2 - n_1) + 2(2 - N) \mp 2(r_{\hat{j}} + N - \hat{j} + 1) \tag{2.15b}$$

$$d = d^{\pm}_{3\hat{j}} \doteq \tfrac{1}{2}(n_3 - 2n_2 - n_1) + 2(1 - N) \mp 2(r_{\hat{j}} + N - \hat{j} + 1) \tag{2.15c}$$

$$d = d^{\pm}_{4\hat{j}} \doteq -\tfrac{1}{2}(n_1 + 2n_2 + 3n_3) - 2N \mp 2(r_{\hat{j}} + N - \hat{j} + 1). \tag{2.15d}$$

For future use we note the following relations:

$$\tfrac{1}{2}\big(d^-_{i\hat{j}} - d^-_{k\hat{\ell}}\big) = n_i + \cdots + n_{k-1} + k - i + \hat{\ell} - \hat{j} + a_{\hat{j}} + \cdots + a_{\hat{\ell}-1} > 0$$
$$i \leqslant k \quad \hat{j} \leqslant \hat{\ell} \quad i\hat{j} \neq k\hat{\ell} \tag{2.16a}$$

$$\tfrac{1}{2}\big(d^+_{i\hat{j}} - d^+_{k\hat{\ell}}\big) = n_i + \cdots + n_{k-1} + k - i + \hat{j} - \hat{\ell} + a_{\hat{\ell}} + \cdots + a_{\hat{j}-1} > 0$$
$$i \leqslant k \quad \hat{j} \geqslant \hat{\ell} \quad i\hat{j} \neq k\hat{\ell} \tag{2.16b}$$

$$\tfrac{1}{2}\big(d^-_{i\hat{j}} - d^+_{k\hat{\ell}}\big) = n_i + \cdots + n_{k-1} + k - i + 2N - \hat{j} - \hat{\ell} + r_{\hat{j}} + r_{\hat{\ell}} + 2 > 0 \qquad i \leqslant k \tag{2.16c}$$

which introduce some partial ordering between the quantities $d^{\pm}_{i\hat{j}}$ of which the essential would turn out to be the following:

$$d^-_{11} > d^-_{21} > d^-_{31} > d^-_{41}. \tag{2.17}$$

The four values in (2.17) play a special role in the unitarity formulation. The value d^-_{11} is the biggest among all $d^{\pm}_{ij}$; it is called 'the first reduction point' in [38].

2.6. *Shapovalov form and unitarity*

The Shapovalov form is a bilinear $\mathbb{C}$-valued form on Verma modules [46]. We also need the involutive antiautomorphism ω of $U(\mathcal{G}^+)$ which will provide the real form we are interested in. Thus, an adaptation of the Shapovalov form suitable for our purposes is defined (as in [42]) as follows:

$$\begin{gathered}(u, u') = (pv_0, p'v_0) \equiv (v_0, \omega(p)p'v_0) = (\omega(p')pv_0, v_0) \\ u = pv_0 \qquad u' = p'v_0 \qquad p, p' \in U(\mathcal{G}^+) \qquad u, u' \in V^{\Lambda}\end{gathered} \tag{2.18}$$

supplemented by the normalization condition $(v_0, v_0) = 1$. The norms squared of the states would be denoted by

$$\|u\|^2 \equiv (u, u). \tag{2.19}$$

We suppose that we consider representations which are unitary when restricted to the even part $\mathcal{G}^+_{\bar{0}}$. This is justified *a posteriori* since (as in the $D = 4$ case [40, 42]) the unitary bounds of the even part are weaker than the supersymmetric ones [47]. Thus, as in [40, 42] we shall factorize the even part and we shall consider only the states created by the action of the odd generators, i.e. $\mathcal{F}^{\Lambda} = \big(U(\mathcal{G}^+)/U(\mathcal{G}^+_{\bar{0}})\big)v_0$. We introduce notation $X^+_{i\hat{j}}$ for the odd generator corresponding to the positive root $\epsilon_i - \delta_{\hat{j}}$, and $Y^+_{i\hat{j}}$ shall correspond to $\epsilon_i + \delta_{\hat{j}}$. Since the odd generators are Grassmann there are only 2^{8N} states in $\mathcal{F}$ and choosing an ordering we give these states explicitly as follows:

$$\Psi_{\bar{\varepsilon}\bar{\nu}} = \left(\prod_{i=1}^{4}\big(Y^+_{i1}\big)^{\varepsilon_{i1}}\right)\cdots\left(\prod_{i=1}^{4}\big(Y^+_{iN}\big)^{\varepsilon_{iN}}\right)\left(\prod_{i=1}^{4}\big(X^+_{iN}\big)^{\nu_{iN}}\right)\cdots\left(\prod_{i=1}^{4}\big(X^+_{i1}\big)^{\nu_{i1}}\right)v_0$$
$$\varepsilon_{i\hat{j}}, \nu_{i\hat{j}} = 0, 1 \tag{2.20}$$

where $\bar{\varepsilon}$, $\bar{\nu}$, denote the set of all $\varepsilon_{i\hat{\jmath}}$, $\nu_{i\hat{\jmath}}$, resp. For future use we give the notation for the number of Ys and Xs:

$$\varepsilon \equiv \sum_{i=1}^{4} \sum_{\hat{\jmath}=1}^{N} \varepsilon_{i\hat{\jmath}} \qquad \nu \equiv \sum_{i=1}^{4} \sum_{\hat{\jmath}=1}^{N} \nu_{i\hat{\jmath}} \tag{2.21}$$

and through them for the *level* ℓ:

$$\ell\,(\Psi_{\bar{\varepsilon}\bar{\nu}}) = \varepsilon + \nu. \tag{2.22}$$

2.7. Explicit realization of the basis of osp(8/2N)

To proceed further we need the explicit realization of the generators of $osp(8/2N)$. It is obtained from the standard one of [44] by applying a unitary transformation done in order to bring the Cartan subalgebra in diagonal form. The matrices are $(8+2N) \times (8+2N)$ and are in standard supermatrix form, i.e. the even are of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

and the odd of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

The description is done very conveniently in terms of the matrices $E_{AB} \in gl(8/2N, \mathbb{C})$, $A, B = 1, \ldots, 8+2N$. Fix A, B, then the matrix E_{AB} has only non-zero entry, equal to 1, at the intersection of the Ath row and Bth column.

Then the generators H_A are given by

$$\begin{aligned}
&H_j = E_{jj} - E_{j+1,j+1} - E_{j+4,j+4} + E_{j+5,j+5} \qquad j = 1,2,3 \\
&H_4 = E_{33} + E_{44} - E_{77} - E_{88} \\
&H_{4+\hat{\jmath}} = E_{8+\hat{\jmath},8+\hat{\jmath}} - E_{9+\hat{\jmath},9+\hat{\jmath}} - E_{8+N+\hat{\jmath},8+N+\hat{\jmath}} + E_{9+N+\hat{\jmath},9+N+\hat{\jmath}} \qquad \hat{\jmath} = 1,\ldots,N-1 \\
&H_{4+N} = E_{8+N,8+N} - E_{8+2N,8+2N}
\end{aligned} \tag{2.23}$$

the basis of $\mathcal{G}^+$—enumerated by the corresponding roots—is

$$\begin{aligned}
&L^+_{ij} = E_{ij} - E_{4+j,4+i} && \text{roots}: \epsilon_i - \epsilon_j && i < j \\
&P^+_{ij} = E_{i,4+j} - E_{j,4+i} && \text{roots}: \epsilon_i + \epsilon_j && i < j \\
&T^+_{\hat{\imath}\hat{\jmath}} = E_{8+\hat{\imath},8+\hat{\jmath}} - E_{8+N+\hat{\jmath},8+N+\hat{\imath}} && \text{roots}: \delta_{\hat{\imath}} - \delta_{\hat{\jmath}} && \hat{\imath} < \hat{\jmath} \\
&R^+_{\hat{\imath}\hat{\jmath}} = E_{8+\hat{\imath},8+N+\hat{\jmath}} + E_{8+\hat{\jmath},8+N+\hat{\imath}} && \text{roots}: \delta_{\hat{\imath}} + \delta_{\hat{\jmath}} && \hat{\imath} < \hat{\jmath} \\
&R^+_{\hat{\imath}} = E_{8+\hat{\imath},8+N+\hat{\imath}} && \text{roots}: 2\delta_{\hat{\imath}} \\
&X^+_{i\hat{\jmath}} = E_{i,8+\hat{\jmath}} + E_{8+N+\hat{\jmath},4+i} && \text{roots}: \epsilon_i - \delta_{\hat{\jmath}} \\
&Y^+_{i\hat{\jmath}} = E_{i,8+N+\hat{\jmath}} - E_{8+\hat{\jmath},4+i} && \text{roots}: \epsilon_i + \delta_{\hat{\jmath}}
\end{aligned} \tag{2.24}$$

while the basis of $\mathcal{G}^-$ is

$$\begin{aligned}
&L^-_{ij} = E_{ij} - E_{4+j,4+i} && \text{roots}: \epsilon_i - \epsilon_j && i > j \\
&P^-_{ij} = E_{4+j,i} - E_{4+i,j} && \text{roots}: -(\epsilon_i + \epsilon_j) && i < j \\
&T^-_{\hat{\imath}\hat{\jmath}} = E_{8+\hat{\imath},8+\hat{\jmath}} - E_{8+N+\hat{\jmath},8+N+\hat{\imath}} && \text{roots}: \delta_{\hat{\imath}} - \delta_{\hat{\jmath}} && \hat{\imath} > \hat{\jmath} \\
&R^-_{\hat{\imath}\hat{\jmath}} = E_{8+N+\hat{\imath},8+\hat{\jmath}} + E_{8+N+\hat{\jmath},8+\hat{\imath}} && \text{roots}: -(\delta_{\hat{\imath}} + \delta_{\hat{\jmath}}) && \hat{\imath} < \hat{\jmath} \\
&R^-_{\hat{\imath}} = E_{8+N+\hat{\imath},8+\hat{\imath}} && \text{roots}: -2\delta_{\hat{\imath}} \\
&X^-_{i\hat{\jmath}} = E_{4+\hat{\imath},8+N+\hat{\jmath}} - E_{8+\hat{\jmath},i} && \text{roots}: -\epsilon_i + \delta_{\hat{\jmath}} \\
&Y^-_{i\hat{\jmath}} = E_{4+i,8+\hat{\jmath}} + E_{8+N+\hat{\jmath},i} && \text{roots}: -(\epsilon_i + \delta_{\hat{\jmath}}).
\end{aligned} \tag{2.25}$$

From the explicit matrix realization above one easily obtains all commutation relations. We shall write down only some more important ones:

$$\left[X^+_{i\hat{\jmath}}, X^-_{i\hat{\jmath}}\right]_+ = -E_{ii} + E_{4+i,4+i} - E_{8+\hat{\jmath},8+\hat{\jmath}} + E_{8+N+\hat{\jmath},8+N+\hat{\jmath}} = -\hat{H}_i - \tilde{H}_{\hat{\jmath}} \tag{2.26a}$$

$$\left[Y^+_{i\hat{\jmath}}, Y^-_{i\hat{\jmath}}\right]_+ = E_{ii} - E_{4+i,4+i} - E_{8+\hat{\jmath},8+\hat{\jmath}} + E_{8+N+\hat{\jmath},8+N+\hat{\jmath}} = \hat{H}_i - \tilde{H}_{\hat{\jmath}} \tag{2.26b}$$

where we have introduced notation for an alternative basis of $\mathcal{H}$ which actually is used in the calculation of scalar products

$$\hat{H}_i \equiv E_{ii} - E_{4+i,4+i} \qquad i = 1, \ldots, 4 \tag{2.26c}$$

$$\tilde{H}_{\hat{\jmath}} \equiv E_{8+\hat{\jmath},8+\hat{\jmath}} - E_{8+N+\hat{\jmath},8+N+\hat{\jmath}} \qquad \hat{\jmath} = 1, \ldots, N. \tag{2.26d}$$

In particular, we shall use continuously

$$\left[\hat{H}_k, X^+_{i\hat{\jmath}}\right] = \delta_{ki} X^+_{i\hat{\jmath}} \tag{2.27a}$$

$$\left[\hat{H}_k, Y^+_{i\hat{\jmath}}\right] = \delta_{ki} Y^+_{i\hat{\jmath}} \tag{2.27b}$$

$$\left[\tilde{H}_{\hat{\ell}}, X^+_{i\hat{\jmath}}\right] = -\delta_{\hat{\ell}\hat{\jmath}} X^+_{i\hat{\jmath}} \tag{2.27c}$$

$$\left[\tilde{H}_{\hat{\ell}}, Y^+_{i\hat{\jmath}}\right] = \delta_{\hat{\ell}\hat{\jmath}} Y^+_{i\hat{\jmath}}. \tag{2.27d}$$

We also give the generators $\hat{H}_A$ in terms of H_A

$$\begin{aligned}
\hat{H}_1 &= H_1 + H_2 + \tfrac{1}{2}(H_4 + H_3) \\
\hat{H}_2 &= H_2 + \tfrac{1}{2}(H_4 + H_3) \\
\hat{H}_3 &= \tfrac{1}{2}(H_4 + H_3) \\
\hat{H}_4 &= \tfrac{1}{2}(H_4 - H_3) \\
\tilde{H}_{\hat{\jmath}} &= H_{4+\hat{\jmath}} + \cdots + H_{4+N} \qquad \hat{\jmath} = 1, \ldots, N.
\end{aligned} \tag{2.28}$$

3. Unitarity

In this section we state our main result (in the theorem) on the lowest weight (positive energy) UIRs and give the proof of necessity in general and proof of sufficiency at generic points (the reduction points are dealt with in the next section).

3.1. Calculation of some norms

In this subsection we show how to use the form (2.18) to calculate the norms of the states from $\mathcal{F}$.

We first need explicitly the conjugation ω on the odd generators:

$$\omega\left(X^+_{i\hat{\jmath}}\right) = -X^-_{i\hat{\jmath}} \qquad \omega\left(Y^+_{i\hat{\jmath}}\right) = Y^-_{i\hat{\jmath}}. \tag{3.1}$$

(In matrix notation this would follow from: $\omega(E_{i,8+\hat{\jmath}}) = E_{8+\hat{\jmath},i}, \omega(E_{i+4,8+\hat{\jmath}}) = -E_{8+\hat{\jmath},i+4}$.)

We give now explicitly the norms of the one-particle states from $\mathcal{F}$ introducing also notation for future use:

$$\begin{aligned}
x_{i\hat{\jmath}} &\equiv \left\|X^+_{i\hat{\jmath}} v_0\right\|^2 = \left(X^+_{i\hat{\jmath}} v_0, X^+_{i\hat{\jmath}} v_0\right) = -\left(v_0, X^-_{i\hat{\jmath}} X^+_{i\hat{\jmath}} v_0\right) \\
&= (v_0, (\hat{H}_i + \tilde{H}_{\hat{\jmath}}) v_0) = \Lambda(\hat{H}_i + \tilde{H}_{\hat{\jmath}})
\end{aligned} \tag{3.2a}$$

$$y_{i\hat{\jmath}} \equiv \left\| Y^+_{i\hat{\jmath}} v_0 \right\|^2 = \left(Y^+_{i\hat{\jmath}} v_0, Y^+_{i\hat{\jmath}} v_0 \right) = \left(v_0, Y^-_{i\hat{\jmath}} Y^+_{i\hat{\jmath}} v_0 \right)$$
$$= (v_0, (\hat{H}_i - \tilde{H}_{\hat{\jmath}}) v_0) = \Lambda(\hat{H}_i - \tilde{H}_{\hat{\jmath}}). \tag{3.2b}$$

Using (2.28), (2.8) and (2.9) we get

$$x_{i\hat{\jmath}} = (\Lambda, \epsilon_i - \delta_{\hat{\jmath}}) = \tfrac{1}{2}\left(d - d^-_{i\hat{\jmath}}\right) + 5 - i - \hat{\jmath} \tag{3.3a}$$

$$y_{i\hat{\jmath}} = (\Lambda, \epsilon_i + \delta_{\hat{\jmath}}) = \tfrac{1}{2}\left(d - d^+_{i\hat{\jmath}}\right) + 3 - i + \hat{\jmath} - 2N. \tag{3.3b}$$

And we note:

$$x_{i+1,\hat{\jmath}} - x_{i\hat{\jmath}} = \tfrac{1}{2}\left(d^-_{i\hat{\jmath}} - d^-_{i+1,\hat{\jmath}}\right) - 1 = n_i \geqslant 0 \tag{3.4a}$$

$$x_{i,\hat{\jmath}+1} - x_{i\hat{\jmath}} = \tfrac{1}{2}\left(d^-_{i\hat{\jmath}} - d^-_{i,\hat{\jmath}+1}\right) - 1 = a_{\hat{\jmath}} \geqslant 0 \tag{3.4b}$$

$$y_{i+1,\hat{\jmath}} - y_{i\hat{\jmath}} = \tfrac{1}{2}\left(d^+_{i\hat{\jmath}} - d^+_{i+1,\hat{\jmath}}\right) - 1 = n_i \geqslant 0 \tag{3.4c}$$

$$y_{i\hat{\jmath}} - y_{i,\hat{\jmath}+1} = \tfrac{1}{2}\left(d^+_{i,\hat{\jmath}+1} - d^+_{i\hat{\jmath}}\right) - 1 = a_{\hat{\jmath}} \geqslant 0 \tag{3.4d}$$

$$y_{i,\hat{\ell}} - x_{i\hat{\jmath}} = \tfrac{1}{2}\left(d^-_{i\hat{\jmath}} - d^+_{i,\hat{\ell}}\right) + \hat{\jmath} + \hat{\ell} - 2N - 2 = r_{\hat{\ell}} + r_{\hat{\jmath}} \geqslant 0. \tag{3.4e}$$

Thus, x_{11} is the smallest among all $x_{i\hat{\jmath}}$ and $y_{i\hat{\jmath}}$.

3.2. Statement of main result and proof of necessity

In this subsection we state our main result in the theorem and give the proof of necessity via two propositions (1 and 2).

First we give the norms which actually determine all of the unitarity conditions. In order to simplify the exposition we shall also use the notation

$$X^+_j \equiv X^+_{j1} \qquad x_j \equiv x_{j1}. \tag{3.5}$$

We note in these terms a subset of (3.4a)

$$x_{i+1} - x_i = n_i \geqslant 0. \tag{3.4a'}$$

Next we calculate

$$\left\| X^+_j X^+_k v_0 \right\|^2 = (x_j - 1) x_k \qquad j < k \tag{3.6a}$$

$$\left\| X^+_j X^+_k X^+_\ell v_0 \right\|^2 = (x_j - 2)(x_k - 1) x_\ell \qquad j < k < \ell \tag{3.6b}$$

$$\left\| X^+_1 X^+_2 X^+_3 X^+_4 v_0 \right\|^2 = (x_1 - 3)(x_2 - 2)(x_3 - 1) x_4. \tag{3.6c}$$

The norms (3.2a) and (3.6) are all strictly positive iff $x_j > 4 - j$, $j = 1, 2, 3, 4$, which are all fulfilled if $x_1 > 3$, since x_1 is the smallest among the x_j. Thus, these norms are strictly positive iff

$$x_1 > 3 \Longleftrightarrow d > d^-_{11}. \tag{3.7}$$

It turns out that this restriction is sufficient to guarantee unitarity of the whole representation. This is not unexpected: in all cases studied so far it was always so that if d is bigger than the first odd reduction point then the module is unitary.

Of course, the condition (3.7) is not necessary for unitarity. On the experience so far it is expected that when d is equal to some of the reducibility values then unitarity is also possible, though in these cases there would be some conditions on the representation parameters, and one has to factor out the resulting zero-norm states. Now we can formulate the main result:

Theorem. *All positive energy unitary irreducible representations of the conformal superalgebra* $osp(8^*/2N)$ *characterized by the signature* χ *in (2.1) are obtained for real* d *and are given in the following list:*

$$d \geqslant d_{11}^- = \tfrac{1}{2}(3n_1 + 2n_2 + n_3) + 2r_1 + 6 \qquad \text{no restrictions on } n_j \qquad (3.8a)$$
$$d = d_{21}^- = \tfrac{1}{2}(n_3 + 2n_2) + 2r_1 + 4 \qquad n_1 = 0 \qquad (3.8b)$$
$$d = d_{31}^- = \tfrac{1}{2}n_3 + 2r_1 + 2 \qquad n_1 = n_2 = 0 \qquad (3.8c)$$
$$d = d_{41}^- = 2r_1 \qquad n_1 = n_2 = n_3 = 0. \qquad (3.8d)$$

Remark 1. For $N = 1, 2$ the theorem was conjectured by Minwalla [43], except that he conjectured unitarity also for the open interval (d_{31}^-, d_{21}^-) with conditions on n_j as in (3.8c). We should note that this conjecture could be reproduced neither by methods of conformal field theory [18], nor by the oscillator method [48] (cf [43]), and thus was in doubt. To compare with the notations of [43] one should use the following substitutions: $n_1 = h_2 - h_3, n_2 = h_1 - h_2, n_3 = h_2 + h_3, r_1 = k$, and h_j are all integer or all half-integer. The fact that $n_j \geqslant 0$ for $j = 1, 2, 3$ translates into: $h_1 \geqslant h_2 \geqslant |h_3|$, i.e. the parameters h_j are of the type often used for representations of $so(2N)$ (though usually for $N \geqslant 4$). Note also that the statement of the theorem is arranged in [43] according to the possible values of n_i first and then the possible values of d. To compare with the notation of [18] we use the substitution $(n_1, n_2, n_3) \to (J_3, J_2, J_1)$. Some UIRs at the four exceptional points d_{i1}^- were constructed in [49] by the oscillator method (some of these were identified with Cartan-type signatures like (2.1) in, e.g., [43, 29]). □

The proof of the theorem requires showing that there is unitarity as claimed, i.e. that the conditions are *sufficient*, and that there is no unitarity otherwise, i.e. that the conditions are *necessary*. For the sufficiency we need all norms, but for the necessity part we only need knowledge of a few norms. We give the necessity part in two propositions.

Proposition 1. *There is no unitarity in any of the open intervals:* $(d_{j+1,1}^-, d_{j1}^-)$, $j = 1, 2, 3$, *and if* $d < d_{41}^-$.

Proof.

- Consider d in the open interval (d_{21}^-, d_{11}^-), which means that $3 > x_1 > 2 - n_1$. Consider the norm (3.6c) and using (3.4a) express all x_i in terms of x_1. We have:

$$(x_1 - 3)(x_2 - 2)(x_3 - 1)x_4 = (x_1 - 3)(x_1 + n_1 - 2)(x_1 + n_1 + n_2 - 1)(x_1 + n_1 + n_2 + n_3) \qquad (3.9)$$

  The first term is strictly negative while the other three terms are strictly positive, independent of the values of n_i. Thus, the norm (3.6c) is negative in the open interval (d_{21}^-, d_{11}^-).
- Consider d in the open interval (d_{31}^-, d_{21}^-), which means that $2 > x_1 + n_1 > 1 - n_2$. Consider the norm (3.6b) for $(j, k, \ell) = (1, 3, 4)$ and using (3.4a) express all x_i in terms of x_1. We have

$$(x_1 - 2)(x_3 - 1)x_4 = (x_1 - 2)(x_1 + n_1 + n_2 - 1)(x_1 + n_1 + n_2 + n_3) \qquad (3.10)$$

The first term is strictly negative while the other two terms are strictly positive, independent of the values of n_i. Thus, the norm of the state $X_1^+X_3^+X_4^+v_0$ is negative in the open interval (d_{32}^-, d_{21}^-).

- Consider d in the open interval (d_{41}^-, d_{31}^-), which means that $1 > x_1 + n_1 + n_2 > -n_3$. Consider the norm (3.6$a$) for $(j, k) = (1, 4)$ and using (3.4a) express all x_i in terms of x_1. We have

$$(x_1 - 1)x_4 = (x_1 - 1)(x_1 + n_1 + n_2 + n_3). \tag{3.11}$$

  The first term is strictly negative while the second is strictly positive, independent of the values of n_i. Thus, the norm of the state $X_1^+X_4^+v_0$ is negative in the open interval (d_{31}^-, d_{21}^-).
- Consider d in the infinite open interval $d < d_{41}^-$. Then the norm of $X_{41}^+v_0$ is negative using (3.3a):

$$x_4 = x_{41} = \tfrac{1}{2}(d - d_{41}^-) < 0.$$

  Thus, the proposition is proved.

□

Thus, we have shown the exclusion of the open intervals in the statement of the theorem. The necessity of the restrictions on n_i in cases b, c, d of the theorem remains to be shown.

Proposition 2. *There is no unitarity in the following cases:*

$$d = d_{21}^- \qquad n_1 > 0 \tag{3.8b'}$$

$$d = d_{31}^- \qquad n_1 + n_2 > 0 \tag{3.8c'}$$

$$d = d_{41}^- \qquad n_1 + n_2 + n_3 > 0. \tag{3.8d'}$$

Proof.

- Let $d = d_{21}^-$ which means $x_2 = 2$ and $x_1 = 2 - n_1$. Consider again the norm of $X_1^+X_3^+X_4^+v_0$ and substitute the value of x_1 in (3.10) to get

$$(x_1 - 2)(x_3 - 1)x_4 = (-n_1)(1 + n_2)(2 + n_2 + n_3). \tag{3.12}$$

  This norm is negative if $n_1 > 0$.
- Let $d = d_{31}^-$ which means $x_3 = 1$ and $x_1 = 1 - n_1 - n_2$. Consider again the norm of $X_1^+X_4^+v_0$ and substitute the value of x_1 in (3.11) to get

$$(x_1 - 1)x_4 = (-n_1 - n_2)(1 + n_3). \tag{3.13}$$

  This norm is negative if $n_1 + n_2 > 0$.
- Let $d = d_{41}^-$ which means $x_4 = 0$ and $x_1 = -n_1 - n_2 - n_3$. But the latter is the norm of $X_1^+v_0$ and it is negative if $n_1 + n_2 + n_3 > 0$.

Thus, the proposition is proved. □

With this we have shown that the conditions of the theorem are *necessary*.
The proof of sufficiency is postponed for the next subsection.

Remark 2. The reader may wonder why the other reducibility points are not playing such an important role as the quartet appearing in the theorem.

First we note that the analogous calculations involving other quartets of operators: $X_{i\hat{\jmath}}^+$ ($\hat{\jmath} \neq 1$ fixed, $i = 1, 2, 3, 4$) or $Y_{i\hat{\jmath}}^+$ ($\hat{\jmath}$ fixed, $i = 1, 2, 3, 4$) give the same results as (3.6) with just replacing $x_i \to x_{i\hat{j}}$ or $x_i \to y_{i\hat{j}}$. This brings the conditions $x_{i\hat{j}} > 4 - i$ or $y_{i\hat{j}} > 4 - i$ which all follow from $x_1 > 3$ because of (3.4). This is related to the fact that d^-_{11} is the largest reduction point.

Further, we may look for the analogue of proposition 1 and we can prove the same results involving $d^-_{i\hat{j}}$ ($\hat{j} \neq 1$ fixed), or $d^+_{i\hat{j}}$ ($\hat{j}$ fixed). However, these results may be relevant only if the exceptional points d^-_{21}, d^-_{31}, d^-_{41}, together with the respective conditions, would happen to be in some of the open intervals defined by some other quartet, which would prove their non-unitarity. The reason that this does not happen is the following.

Let $n_1 = 0$. Then one can easily see that d^-_{11} and d^-_{21} are the two largest reduction points (for $N = 1, 2$ cf [43]), i.e. all other reduction points are smaller, and thus, d^-_{21} cannot be in any open interval defined by some other quartet.

Analogously, for $n_1 = n_2 = 0$ one can easily see that d^-_{11}, d^-_{21} and d^-_{31} are the three largest reduction points (for $N = 1, 2$ cf [43]), and so d^-_{31} cannot be in any open interval defined by some other quartet. Finally, for $n_1 = n_2 = n_3 = 0$ the points d^-_{11}, d^-_{21}, d^-_{31} and d^-_{41} are the four largest reduction points (for $N = 1, 2$ cf [43]), and d^-_{41} cannot be in any open interval defined by some other quartet. □

3.3. General form of the norms and unitarity in the generic case

In this subsection we give the proof of sufficiency of the theorem in the generic case. This requires the general form of the norms. The states are divided into classes and the norms are given for the different cases in propositions 3–7. At the end we finish the proof of sufficiency utilizing these propositions.

To present the general formulae for the norms we first we divide the states into factorizable and unfactorizable as follows. Let the first generator in $\Psi_{\bar{\varepsilon}\bar{\nu}}$ be $Y^+_{i\hat{j}}$, i.e. $\Psi_{\bar{\varepsilon}\bar{\nu}} = Y^+_{i\hat{j}} \cdots v_0$. Then $\Psi_{\bar{\varepsilon}\bar{\nu}}$ is called *factorizable* if the following three statements hold:

$$\epsilon_{k\hat{j}}\epsilon_{i\hat{\ell}} = 0 \quad \text{or} \quad \epsilon_{k\hat{\ell}} = 1 \qquad \text{for all pairs } (k, \hat{\ell}) \text{ such that} : k > i, \hat{\ell} > \hat{j} \tag{3.14a}$$

$$\epsilon_{k\hat{j}}\nu_{i\hat{\ell}} = 0 \quad \text{or} \quad \nu_{k\hat{\ell}} = 1 \qquad \text{for all } k > \imath, \text{ and all } \hat{\ell} \tag{3.14b}$$

$$\epsilon_{i\hat{\ell}}\nu_{j\hat{\ell}} = 0 \quad \text{or} \quad \nu_{j\hat{j}} = 1 \qquad \text{for all } \hat{\ell} > \hat{j}, \text{ and all } j \tag{3.14c}$$

If the first generator in $\Psi_{\bar{\varepsilon}\bar{\nu}}$ is $X^+_{i\hat{j}}$, so that $\bar{\varepsilon} = 0$, then $\Psi_{0,\bar{\nu}}$ is called *factorizable* if the following statement holds:

$$\nu_{j\hat{j}}\nu_{i\hat{\ell}} = 0 \text{ or } \nu_{j\hat{\ell}} = 1 \qquad \text{for all pairs } (j, \hat{\ell}) \text{ such that} : j > i, \hat{\ell} < \hat{j} \tag{3.15}$$

Our first result on the norms is

Proposition 3. *For factorizable states starting with $X^+_{i\hat{j}}$ the following relation holds:*

$$\begin{aligned} &\|\Psi_{0,\bar{\nu}}\|^2 = (x_{i\hat{j}} + \tilde{\nu}_{i\hat{j}})\|\Psi_{0,\bar{\nu}'}\|^2 \\ &\tilde{\nu}_{i\hat{j}} = \nu_{i,\hat{j}-1} + \cdots + \nu_{i,1} - \nu_{i+1,\hat{j}} - \cdots - \nu_{4,\hat{j}} \\ &\nu'_{j\hat{\ell}} = \nu_{j\hat{\ell}} - \delta_{ji}\delta_{\hat{\ell}\hat{j}}. \end{aligned} \tag{3.16}$$

For factorizable states starting with $Y^+_{i\hat{j}}$ the following relation holds:

$$\begin{aligned} &\|\Psi_{\bar{\varepsilon},\bar{\nu}}\|^2 = (y_{i\hat{j}} + \tilde{\varepsilon}_{i\hat{j}} + \nu_i + \hat{\nu}_{\hat{j}})\|\Psi_{\bar{\varepsilon}'\bar{\nu}}\|^2 \\ &\tilde{\varepsilon}_{i\hat{j}} = \varepsilon_{i,\hat{j}+1} + \cdots + \varepsilon_{i,N} - \varepsilon_{i+1,\hat{j}} - \cdots - \varepsilon_{4,\hat{j}} \\ &\nu_i = \nu_{i,1} + \cdots + \nu_{i,N}, \qquad \hat{\nu}_{\hat{j}} = \nu_{1,\hat{j}} + \cdots + \nu_{4,\hat{j}} \\ &\varepsilon'_{j\hat{\ell}} = \varepsilon_{j\hat{\ell}} - \delta_{ji}\delta_{\hat{\ell}\hat{j}}. \end{aligned} \tag{3.17}$$

Proof. We start with (3.16). Clearly, $\Psi_{0,\bar{\nu}} = X^+_{i\hat{\jmath}}\Psi_{0,\bar{\nu}'}$. Then the norm squared is

$$\|\Psi_{0,\bar{\nu}}\|^2 = \left(X^+_{i\hat{\jmath}}\Psi_{0,\bar{\nu}'}, X^+_{i\hat{\jmath}}\Psi_{0,\bar{\nu}'}\right) = -\left(\Psi_{0,\bar{\nu}'}, X^-_{i\hat{\jmath}}X^+_{i\hat{\jmath}}\Psi_{0,\bar{\nu}'}\right) \tag{3.18a}$$

$$= -\left(\Psi_{0,\bar{\nu}'}, \left(-X^+_{i\hat{\jmath}}X^-_{i\hat{\jmath}} - \hat{H}_i - \tilde{H}_{\hat{\jmath}}\right)\Psi_{0,\bar{\nu}'}\right) \tag{3.18b}$$

$$= (\Psi_{0,\bar{\nu}'}, (\hat{H}_i + \tilde{H}_{\hat{\jmath}})\Psi_{0,\bar{\nu}'}) \tag{3.18c}$$

$$= (\Lambda(\hat{H}_i + \tilde{H}_{\hat{\jmath}}) + \tilde{\nu}_{i\hat{\jmath}})(\Psi_{0,\bar{\nu}'}, \Psi_{0,\bar{\nu}'}) \tag{3.18d}$$

$$= (x_{i\hat{\jmath}} + \tilde{\nu}_{i\hat{\jmath}})\|\Psi_{0,\bar{\nu}'}\|^2. \tag{3.18e}$$

Note that the term $X^+_{i\hat{\jmath}}X^-_{i\hat{\jmath}}$ in (3.18b) gives no contribution: due to conditions (3.15) the operator $X^-_{i\hat{\jmath}}$ anticommutes with the operators in $\Psi_{0,\bar{\nu}'}$ or produces terms such as: $\left(X^+_{j\hat{\ell}}\right)^2 = 0$, thus it reaches v_0 without additional terms. Moving the operator $\hat{H}_i + \tilde{H}_{\hat{\jmath}}$ through $\Psi_{0,\bar{\nu}'}$ produces the addition $\tilde{\nu}_{i\hat{\jmath}}$—the terms $\nu_{i,\hat{\jmath}-1} + \cdots + \nu_{i,1}$ are due to (2.27a), and the terms $-\nu_{i+1,\hat{\jmath}} - \cdots - \nu_{4,\hat{\jmath}}$ are due to (2.27c). Analogously, we consider (3.17). Clearly, $\Psi_{\bar{\varepsilon},\bar{\nu}} = Y^+_{i\hat{\jmath}}\Psi_{\bar{\varepsilon}',\bar{\nu}}$. The norm squared is

$$\|\Psi_{\bar{\varepsilon}\bar{\nu}}\|^2 = \left(Y^+_{i\hat{\jmath}}\Psi_{\bar{\varepsilon}',\bar{\nu}}, Y^+_{i\hat{\jmath}}\Psi_{\bar{\varepsilon}',\bar{\nu}}\right) = \left(\Psi_{\bar{\varepsilon}',\bar{\nu}}, Y^-_{i\hat{\jmath}}Y^+_{i\hat{\jmath}}\Psi_{\bar{\varepsilon}',\bar{\nu}}\right) \tag{3.19a}$$

$$= \left(\Psi_{\bar{\varepsilon}',\bar{\nu}}, \left(-Y^+_{i\hat{\jmath}}Y^-_{i\hat{\jmath}} + \hat{H}_i - \tilde{H}_{\hat{\jmath}}\right)\Psi_{\bar{\varepsilon}',\bar{\nu}}\right) \tag{3.19b}$$

$$= (\Psi_{\bar{\varepsilon}',\bar{\nu}}, (\hat{H}_i - \tilde{H}_{\hat{\jmath}})\Psi_{\bar{\varepsilon}',\bar{\nu}}) \tag{3.19c}$$

$$= (\Lambda(\hat{H}_i - \tilde{H}_{\hat{\jmath}}) + \tilde{\varepsilon}_{i\hat{\jmath}} + \nu_i + \hat{\nu}_{\hat{\jmath}})(\Psi_{\bar{\varepsilon}',\bar{\nu}}, \Psi_{\bar{\varepsilon}',\bar{\nu}}) \tag{3.19d}$$

$$= (y_{i\hat{\jmath}} + \tilde{\varepsilon}_{i\hat{\jmath}} + \nu_i + \hat{\nu}_{\hat{\jmath}})\|\Psi_{\bar{\varepsilon}',\bar{\nu}}\|^2. \tag{3.19e}$$

Note that to produce the additional terms $\tilde{\varepsilon}_{i\hat{\jmath}} + \nu_i + \hat{\nu}_{\hat{\jmath}}$ we need all of (2.27). □

The states $\Psi_{\bar{\varepsilon}'\bar{\nu}}$ and $\Psi_{0,\bar{\nu}'}$ may still be factorizable and so on. The state $\Psi_{0,\bar{\nu}}$ is called *fully factorizable* if the process of factorization can be repeated ν times. The state $\Psi_{\bar{\varepsilon},\bar{\nu}}$ is called *fully factorizable* if the process of factorization can be repeated ε times and the resulting state $\Psi_{0,\bar{\nu}}$ is fully factorizable.

Our first main result on the norms is

Proposition 4. *The norm of a fully factorizable state $\Psi_{\bar{\varepsilon}\bar{\nu}}$ is given by the following formula:*

$$\|\Psi_{\bar{\varepsilon}\bar{\nu}}\|^2 = \mathcal{N}_{\bar{\varepsilon}\bar{\nu}} \tag{3.20}$$

where

$$\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} = \prod_{i=1}^{4}\prod_{\hat{\jmath}=1}^{N}(y_{i\hat{\jmath}} + \tilde{\varepsilon}_{i\hat{\jmath}} + \nu_i + \tilde{\nu}_{\hat{\jmath}})^{\varepsilon_{ij}}(x_{i\hat{\jmath}} + \tilde{\nu}_{i\hat{\jmath}})^{\nu_{i\hat{\jmath}}} \tag{3.21}$$

Proof. By direct iteration of (3.17) and (3.16). □

Naturally, the norms in (3.6) are special cases of (3.20).

Note that the norm squared of a state is a polynomial in d of degree the level ℓ of the state.

We shall now discuss states which are not fully factorizable. It is enough to consider unfactorizable states, since if a state is factorizable then we apply (3.17) or (3.16) one or more times until we are left with the norm squared of an unfactorizable state. We shall have two propositions, the first of which is

Proposition 5. *Let $\Psi_{0,\bar{\nu}}$ be an unfactorizable state starting with the generator $X^+_{i\hat{\jmath}}$. This means that there are one or more pairs of integers $(k,\hat{\ell})$ so that (3.15) is violated. Let us enumerate the pairs violating (3.15) as*

$$(k_m,\hat{\ell}_{m,n}),\quad i<k_1<\cdots<k_p,\quad \hat{\jmath}>\hat{\ell}_{m,1}>\cdots>\hat{\ell}_{m,q(m)} \tag{3.22}$$

so that the following holds:

$$\nu_{k_m,\hat{\jmath}}=\nu_{i,\hat{\ell}_{m,n}}=1 \text{ and } \nu_{k_m,\hat{\ell}_{m,n}}=0. \tag{3.23}$$

Then the norm of $\Psi_{0,\bar{\nu}}$ is given by the following formula:

$$\|\Psi_{0,\bar{\nu}}\|^2=(x_{i,\hat{\jmath}}+\tilde{\nu}_{i,\hat{\jmath}})\|\Psi_{0,\bar{\nu}'}\|^2-\sum_{m=1}^{p}\sum_{n=1}^{q(m)}\mathcal{R}^{m,n}_{0,\bar{\nu}} \tag{3.24a}$$

$$\mathcal{R}^{1,n}_{0,\bar{\nu}}=\left(\prod_{s=1}^{n-1}\left(x_{i,\hat{\ell}_{1,s}}+\nu_i-\hat{\nu}_{\hat{\ell}_{1,s}}-s+1\right)\right)\|\Psi_{0,\bar{\nu}^{1,n}}\|^2 \tag{3.24b}$$

$$\nu^{1,n}_{i\hat{\jmath}}=\nu^{1,n}_{k_1,\hat{\jmath}}=\nu^{1,n}_{i,\hat{\ell}_{1,1}}=\cdots=\nu^{1,n}_{i,\hat{\ell}_{1,n}}=0\qquad \nu^{1,n}_{k_1,\hat{\ell}_{1,n}}=1\qquad (\textit{the rest of } \nu^{1,n}_{k,\hat{\ell}} \textit{ are as } \nu_{k,\hat{\ell}})$$

$$\mathcal{R}^{2,n}_{0,\bar{\nu}}=\left(x_{k_1,\hat{\jmath}}+\nu_{k_1}-\hat{\nu}_{\hat{\jmath}}\right)\left(\prod_{s=1}^{n-1}\left(x_{i,\hat{\ell}_{2,s}}+\nu_i-\hat{\nu}_{\hat{\ell}_{2,s}}-s+1\right)\right)\|\Psi_{0,\bar{\nu}^{2,n}}\|^2 \tag{3.24c}$$

$$\nu^{2,n}_{i\hat{\jmath}}=\nu^{2,n}_{k_1,\hat{\jmath}}=\nu^{2,n}_{k_2,\hat{\jmath}}=\nu^{2,n}_{i,\hat{\ell}_{2,1}}=\cdots=\nu^{2,n}_{i,\hat{\ell}_{2,n}}=0\qquad \nu^{2,n}_{k_2,\hat{\ell}_{2,n}}=1$$
$$(\textit{the rest of } \nu^{2,n}_{k,\hat{\ell}} \textit{ are as } \nu_{k,\hat{\ell}})$$

$$\mathcal{R}^{3,n}_{0,\bar{\nu}}=\left(x_{k_1,\hat{\jmath}}+\nu_{k_1}-\hat{\nu}_{\hat{\jmath}}\right)\left(x_{k_2,\hat{\jmath}}+\nu_{k_2}-\hat{\nu}_{\hat{\jmath}}+1\right)\left(\prod_{s=1}^{n-1}\left(x_{i,\hat{\ell}_{3,s}}+\nu_i-\hat{\nu}_{\hat{\ell}_{3,s}}-s+1\right)\right)\|\Psi_{0,\bar{\nu}^{3,n}}\|^2 \tag{3.24d}$$

$$\nu^{3,n}_{i\hat{\jmath}}=\nu^{3,n}_{k_1,\hat{\jmath}}=\nu^{3,n}_{k_2,\hat{\jmath}}=\nu^{3,n}_{k_3,\hat{\jmath}}=\nu^{3,n}_{i,\hat{\ell}_{3,1}}=\cdots=\nu^{3,n}_{i,\hat{\ell}_{3,n}}=0\qquad \nu^{3,n}_{k_3,\hat{\ell}_{3,n}}=1$$
$$(\textit{the rest of } \nu^{3,n}_{k,\hat{\ell}} \textit{ are as } \nu_{k,\hat{\ell}})$$

Proof. The reason for the counterterms is in the transmutation of generators which happens for every pair from (3.22), (3.23) by the following mechanism. Let us take one such pair for fixed (m,n). This means that $\Psi_{0,\bar{\nu}}$ contains the operators

$$\Psi_{0,\bar{\nu}}=X^+_{i\hat{\jmath}}\ldots X^+_{k_m,\hat{\jmath}}\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0 \tag{3.25a}$$

and its norm squared is

$$\begin{aligned}\|\Psi_{0,\bar{\nu}}\|^2&=\left(X^+_{i\hat{\jmath}}\ldots X^+_{k_m,\hat{\jmath}}\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0,X^+_{i\hat{\jmath}}\ldots X^+_{k_m,\hat{\jmath}}\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0\right)\\&=(-1)^{\nu}\left(v_0,\ldots X^-_{i,\hat{\ell}_{m,n}}\ldots X^-_{k_m,\hat{\jmath}}\ldots X^-_{i\hat{\jmath}}X^+_{i\hat{\jmath}}\ldots X^+_{k_m,\hat{\jmath}}\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0\right).\end{aligned} \tag{3.25b}$$

Further we shall give only the term of $\|\Psi_{0,\bar{\nu}}\|^2$ which will turn into the discussed counterterm

$$\begin{aligned}\|\Psi_{0,\bar{\nu}}\|^2&\approx(-1)^{\nu+1}\left(v_0,\ldots X^-_{i,\hat{\ell}_{m,n}}\ldots X^-_{k_m,\hat{\jmath}}\ldots X^+_{i\hat{\jmath}}X^-_{i\hat{\jmath}}\ldots X^+_{k_m,\hat{\jmath}}\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0\right)\\&\approx(-1)^{\nu+1}\left(v_0,\ldots X^-_{i,\hat{\ell}_{m,n}}\ldots X^-_{k_m,\hat{\jmath}}X^+_{i\hat{\jmath}}\ldots X^-_{i\hat{\jmath}}X^+_{k_m,\hat{\jmath}}\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0\right)\\&=(-1)^{\nu+1}\left(v_0,\ldots X^-_{i,\hat{\ell}_{m,n}}\ldots\left(-X^+_{i\hat{\jmath}}X^-_{k_m,\hat{\jmath}}-L^+_{i,k_m}\right)\cdots\right.\\&\quad\left.\times\cdots\left(-X^+_{k_m,\hat{\jmath}}X^-_{i\hat{\jmath}}-L^-_{k_m,i}\right)\ldots X^+_{i,\hat{\ell}_{m,n}}\ldots v_0\right)\end{aligned} \tag{3.25c}$$

$$\approx(-1)^{\nu+1}\left(v_0,\ldots X^-_{i,\hat{\ell}_{m,n}}L^+_{i,k_m}\ldots L^-_{k_m,i}X^+_{i,\hat{\ell}_{m,n}}\ldots v_0\right) \tag{3.25d}$$

$$= (-1)^{\nu+1}\big(v_0, \ldots \big(L^+_{i,k_m} X^-_{i,\hat\ell_{m,n}} + X^-_{k_m,\hat\ell_{m,n}}\big) \ldots \times \ldots \big(X^+_{i,\hat\ell_{m,n}} L^-_{k_m,i} + X^+_{k_m,\hat\ell_{m,n}}\big) \ldots v_0\big) \tag{3.25e}$$

$$\approx (-1)^{\nu+1}\left(v_0, \ldots X^-_{k_m,\hat\ell_{m,n}} \ldots X^+_{k_m,\hat\ell_{m,n}} \ldots v_0\right) \tag{3.25f}$$

$$= -\left\| \ldots X^+_{k_m,\hat\ell_{m,n}} \ldots v_0 \right\|^2. \tag{3.25g}$$

Thus, we have shown that the norm squared of $\Psi_{0,\bar\nu}$ contains a term which is the norm squared (with sign 'minus'—hence the word 'counterterm') of a state obtained from $\Psi_{0,\bar\nu}$ by replacing the operators $X^+_{i\hat\jmath}$, $X^+_{k_m,\hat\jmath}$ and $X^+_{i,\hat\ell_{m,n}}$ by the operator $X^+_{k_m,\hat\ell_{m,n}}$. Note that the latter was not present in $\Psi_{0,\bar\nu}$ due to the condition $\nu_{k_m,\hat\ell_{m,n}} = 0$ in (3.23). Note also that the counterterm state is of level $\nu - 2$ which brings the factor $(-1)^{\nu-2}$ in the passage from (3.25*f*) to (3.25*g*) which together with the factor $(-1)^{\nu+1}$ results in the overall minus sign in (3.25*g*). The described transmutation explains totally only the first counterterm in (3.24*b*) obtained for $(m, n) = (1, 1)$. The other counterterms get additional contributions, in particular, from terms which we neglected in (3.25). For the rest of the counterterms with $(m = 1, n > 1)$ this affects the contributions of the operators $X^+_{i,\hat\ell_{1,s}}$, $s < n$. Analogously, for $m > 1$ this affects in addition the operators $X^+_{k_s,\hat\jmath}$, $s < m$. In all cases, every counterterm is a polynomial in d of degree $\nu - 2$. The overall restrictions on the number of counterterms only remains to be explained. Since $i < 4$, $\hat\jmath > 1$, it follows that $p \leqslant 4 - i \leqslant 3$, $q(m) \leqslant \hat\jmath - 1$. □

Our next main result on the norms is

Proposition 6. *Let $\Psi_{\bar\varepsilon\bar\nu}$ be a unfactorizable state starting with the generator $Y^+_{i\hat\jmath}$. This means that there are one or more pairs of integers $(k, \hat\ell)$ so that (3.14) is violated. Let us enumerate the pairs violating (3.14a) as*

$$(j_m, \hat\jmath_{m,n}) \qquad i < j_1 < \cdots < j_p \qquad \hat\jmath < \hat\jmath_{m,1} < \cdots < \hat\jmath_{m,q(m)} \tag{3.26}$$

(note that $i < 4$, $\hat\jmath < N$, $p \leqslant 4 - i \leqslant 3$, $q(m) \leqslant N - \hat\jmath$) so that the following holds:

$$\varepsilon_{j_m,\hat\jmath} = \varepsilon_{i,\hat\jmath_{m,n}} = 1 \qquad \text{and} \qquad \varepsilon_{j_m,\hat\jmath_{m,n}} = 0 \tag{3.27}$$

Let us enumerate the pairs violating (3.14*b*) as

$$(k_m, \hat k_{m,n}) \qquad i < k_1 < \cdots < k_{p'} \qquad \hat k_{m,1} > \cdots > \hat k_{m,q'(m)} \tag{3.28}$$

(note that $i < 4$, $p' \leqslant 4 - i \leqslant 3$, $q'(m) \leqslant N$) so that the following holds:

$$\varepsilon_{k_m,\hat\jmath} = \nu_{i,\hat k_{m,n}} = 1 \qquad \text{and} \qquad \nu_{k_m,\hat k_{m,n}} = 0 \tag{3.29}$$

Let us enumerate the pairs violating (3.14*c*) as

$$(\ell_m, \hat\ell_{m,n}) \qquad \ell_1 < \cdots < \ell_{p''} \qquad \hat\jmath < \hat\ell_{m,1} < \cdots < \hat\ell_{m,q''(m)} \tag{3.30}$$

(note that $\hat\jmath < N$, $p'' \leqslant 4$, $q''(m) \leqslant N - \hat\jmath$) so that the following holds:

$$\varepsilon_{i,\hat\ell_{m,n}} = \nu_{\ell_m,\hat\ell_{m,n}} = 1 \qquad \text{and} \qquad \nu_{\ell_m,\hat\jmath} = 0. \tag{3.31}$$

Then the norm is given by the following formula:

$$\|\Psi_{\bar\varepsilon\bar\nu}\|^2 = (y_{i,\hat\jmath} + \tilde\varepsilon_{i,\hat\jmath} + \nu_i + \tilde\nu_{\hat\jmath})\|\Psi_{\bar\varepsilon'\bar\nu}\|^2 - \sum_{m=1}^{p}\sum_{n=1}^{q(m)} \mathcal{R}^{m,n}_{\bar\varepsilon,\bar\nu} - \sum_{m=1}^{p'}\sum_{n=1}^{q'(m)} \mathcal{R}'^{m,n}_{\bar\varepsilon,\bar\nu} - \sum_{m=1}^{p''}\sum_{n=1}^{q''(m)} \mathcal{R}''^{m,n}_{\bar\varepsilon,\bar\nu} \tag{3.32a}$$

$$\mathcal{R}^{1,n}_{\bar{\varepsilon},\bar{\nu}} = \left(\prod_{s=1}^{n-1}\left(y_{i,\hat{\jmath}_{1,s}} + \varepsilon_i - \hat{\varepsilon}_{\hat{\jmath}_{1,s}} - s + 1 + \nu_i + \hat{\nu}_{\hat{\jmath}_{1,s}}\right)\right)\|\Psi_{\bar{\varepsilon}^{1,n},\bar{\nu}}\|^2 \tag{3.32b}$$

$$\varepsilon_i = \varepsilon_{i,1} + \cdots + \varepsilon_{i,N} \qquad \hat{\varepsilon}_{\hat{\ell}} = \varepsilon_{1,\hat{\ell}} + \cdots + \varepsilon_{4,\hat{\ell}}$$

$$\varepsilon^{1,n}_{i,\hat{\jmath}} = \varepsilon^{1,n}_{j_1,\hat{\jmath}} = \varepsilon^{1,n}_{i,\hat{\jmath}_{1,1}} = \cdots = \varepsilon^{1,n}_{i,\hat{\jmath}_{1,n}} = 0 \qquad \varepsilon^{1,n}_{j_1,\hat{\jmath}_{1,n}} = 1$$

(the rest of $\varepsilon^{1,n}_{k,\hat{\ell}}$ are as $\varepsilon_{k,\hat{\ell}}$)

$$\mathcal{R}^{2,n}_{\bar{\varepsilon},\bar{\nu}} = \left(y_{j_1,\hat{\jmath}} + \varepsilon_{j_1} - \hat{\varepsilon}_{\hat{\jmath}} + \nu_{j_1} + \hat{\nu}_{\hat{\jmath}}\right)\left(\prod_{s=1}^{n-1}\left(y_{i,\hat{\jmath}_{2,s}} + \varepsilon_i - \hat{\varepsilon}_{\hat{\jmath}_{2,s}} - s + 1 + \nu_i + \hat{\nu}_{\hat{\jmath}_{2,s}}\right)\right)\|\Psi_{\bar{\varepsilon}^{2,n},\bar{\nu}}\|^2 \tag{3.32c}$$

$$\varepsilon^{2,n}_{i,\hat{\jmath}} = \varepsilon^{2,n}_{j_1,\hat{\jmath}} = \varepsilon^{2,n}_{j_2,\hat{\jmath}} = \varepsilon^{2,n}_{i,\hat{\jmath}_{2,1}} = \cdots = \varepsilon^{2,n}_{i,\hat{\jmath}_{2,n}} = 0 \qquad \varepsilon^{2,n}_{j_2,\hat{\jmath}_{2,n}} = 1$$

(the rest of $\varepsilon^{2,n}_{k,\hat{\ell}}$ are as $\varepsilon_{k,\hat{\ell}}$)

$$\begin{aligned}\mathcal{R}^{3,n}_{\bar{\varepsilon},\bar{\nu}} &= \left(y_{j_1,\hat{\jmath}} + \varepsilon_{j_1} - \hat{\varepsilon}_{\hat{\jmath}} + \nu_{j_1} + \hat{\nu}_{\hat{\jmath}}\right)\left(y_{j_2,\hat{\jmath}} + \varepsilon_{j_2} - \hat{\varepsilon}_{\hat{\jmath}} + 1 + \nu_{j_2} + \hat{\nu}_{\hat{\jmath}}\right)\\ &\quad\times\left(\prod_{s=1}^{n-1}\left(y_{i,\hat{\jmath}_{3,s}} + \varepsilon_i - \hat{\varepsilon}_{\hat{\jmath}_{3,s}} - s + 1 + \nu_i + \hat{\nu}_{\hat{\jmath}_{3,s}}\right)\right)\|\Psi_{\bar{\varepsilon}^{3,n},\bar{\nu}}\|^2\end{aligned} \tag{3.32d}$$

$$\varepsilon^{3,n}_{i,\hat{\jmath}} = \varepsilon^{3,n}_{j_1,\hat{\jmath}} = \varepsilon^{3,n}_{j_2,\hat{\jmath}} = \varepsilon^{3,n}_{j_3,\hat{\jmath}} = \varepsilon^{3,n}_{i,\hat{\jmath}_{3,1}} = \cdots = \varepsilon^{3,n}_{i,\hat{\jmath}_{3,n}} = 0 \qquad \varepsilon^{3,n}_{j_3,\hat{\jmath}_{3,n}} = 1$$

(the rest of $\varepsilon^{3,n}_{k,\hat{\ell}}$ are as $\varepsilon_{k,\hat{\ell}}$)

$$\begin{aligned}\mathcal{R}'^{m,n}_{\bar{\varepsilon},\bar{\nu}} &= \left(\prod_{1\leqslant j\leqslant 4}\ \prod_{\substack{\hat{\jmath}\leqslant\hat{m}\leqslant N\\ (j,\hat{m})\neq(i,\hat{\jmath}),(k_m,\hat{\jmath})}}\left(y_{j\hat{m}} + \varepsilon'^{m}_{j} - \hat{\varepsilon}'_{\hat{m}} + \nu'_j + \tilde{\nu}'^{m,n}_{\hat{m}}\right)^{\varepsilon_{j\hat{m}}}\right)\\ &\quad\times\left(\prod_{s=1}^{n-1}\left(x_{i,\hat{k}_{m,s}} + \nu_i - \hat{\nu}_{\hat{k}_{m,s}} - s + 1\right)\right)\|\Psi_{0,\bar{\nu}'^{m,n}}\|^2\end{aligned} \tag{3.32e}$$

$$\varepsilon'^{m}_{j} = \varepsilon_{j,1} + \cdots + \varepsilon_{j,N} - \delta_{ji} - \delta_{j,k_m} \qquad \hat{\varepsilon}'_{\hat{m}} = \varepsilon_{1,\hat{m}} + \cdots + \varepsilon_{4,\hat{m}} - 2\delta_{\hat{m},\hat{\jmath}}$$

$$\nu'_j = \nu_j - \delta_{ij} \qquad \tilde{\nu}'^{m,n}_{\hat{m}} = \tilde{\nu}_{\hat{m}} - \delta_{\hat{m},\hat{k}_{m,n}}$$

$$\nu'^{m,n}_{i,\hat{k}_{m,1}} = \cdots = \nu'^{m,n}_{i,\hat{k}_{m,n}} = 0 \qquad \nu'^{m,n}_{k_m,\hat{k}_{m,n}} = 1$$

(the rest of $\nu'^{m,n}_{k,\hat{\ell}}$ are as $\nu_{k,\hat{\ell}}$)

$$\begin{aligned}\mathcal{R}''^{m,n}_{\bar{\varepsilon},\bar{\nu}'} &= \left(\prod_{1\leqslant j\leqslant 4}\ \prod_{\substack{\hat{\jmath}\leqslant\hat{m}\leqslant N\\ (j,\hat{m})\neq(i,\hat{\jmath}),(i,\hat{\ell}_{m,n})}}\left(y_{j\hat{m}} + \varepsilon''^{m}_{j} - \hat{\varepsilon}''_{\hat{m}} + \nu''_j + \tilde{\nu}''^{m,n}_{\hat{m}}\right)^{\varepsilon_{j\hat{m}}}\right)\\ &\quad\times\left(\prod_{s=1}^{n-1}\left(x_{\hat{\ell}_m,\hat{\ell}_{m,s}} + \nu_{\hat{\ell}_m} - \hat{\nu}_{\hat{\ell}_{m,s}} - s + 1\right)\right)\|\Psi_{0,\bar{\nu}''^{m,n}}\|^2\end{aligned} \tag{3.32f}$$

$$\varepsilon''^{m}_{j} = \varepsilon_{j,1} + \cdots + \varepsilon_{j,N} - 2\delta_{ji} \qquad \hat{\varepsilon}''_{\hat{m}} = \varepsilon_{1,\hat{m}} + \cdots + \varepsilon_{4,\hat{m}} - \delta_{\hat{m},\hat{\jmath}} - \delta_{\hat{m},\hat{\ell}_{m,n}}$$

$$\nu''_j = \nu_j - \delta_{j,\hat{\ell}_m} \qquad \tilde{\nu}''^{m,n}_{\hat{m}} = \tilde{\nu}_{\hat{m}} - \delta_{\hat{m},\hat{\ell}_{m,n}} \qquad \nu''^{m,n}_{\ell_m,\hat{\ell}_{m,1}} = \cdots = \nu''^{m,n}_{\ell_m,\hat{\ell}_{m,n}} = 0$$

$\nu''^{m,n}_{\ell_m,\hat{\jmath}} = 1$ (the rest of $\nu''^{m,n}_{k,\hat{\ell}}$ are as $\nu_{k,\hat{\ell}}$)

The proof of this proposition is analogous to that of proposition 5, though more complicated since there are three possible mechanisms of transmutations corresponding to the three exceptional situations given. Thus, in the case described by (3.26), (3.27) the transmutation is

$$\Psi_{\bar{\varepsilon},\bar{\nu}} = Y^+_{i\hat{\jmath}} \dots Y^+_{j_m,\hat{\jmath}} \dots Y^+_{i,\hat{\jmath}_{m,n}} \dots v_0 \longrightarrow \dots Y^+_{j_m,\hat{\jmath}_{m,n}} \dots v_0. \tag{3.33}$$

In the case described by (3.28), (3.29) the transmutation is

$$\Psi_{\bar{\varepsilon},\bar{\nu}} = Y^+_{i\hat{\jmath}} \dots Y^+_{k_m,\hat{\jmath}} \dots X^+_{i,\hat{k}_{m,n}} \dots v_0 \longrightarrow \dots X^+_{k_m,\hat{k}_{m,n}} \dots v_0. \tag{3.34}$$

In the case described by (3.30), (3.31) the transmutation is

$$\Psi_{\bar{\varepsilon},\bar{\nu}} = Y^+_{i\hat{\jmath}} \dots Y^+_{i,\hat{\ell}_{m,n}} \dots X^+_{\ell_m,\hat{\ell}_{m,n}} \dots v_0 \longrightarrow \dots X^+_{\ell_m,\hat{\jmath}} \dots v_0. \tag{3.35}$$

Note that for $N = 1$ only the cases described by (3.28), (3.29) are possible. Further we proceed as for proposition 5.

Our final main result on the norms is

Proposition 7. *If a state is not fully factorizable then the general expression of its norm is:*

$$\|\Psi_{\bar{\varepsilon}\bar{\nu}}\|^2 = \mathcal{N}_{\bar{\varepsilon}\bar{\nu}} - \mathcal{R}_{\bar{\varepsilon}\bar{\nu}} \tag{3.36}$$

where $\mathcal{R}_{\bar{\varepsilon}\bar{\nu}}$ designates the possible counterterms.

Proof. This follows from propositions 5 and 6. Consider first $\Psi_{0,\bar{\nu}'}$ from proposition 5. If it is fully factorizable, then (3.36) follows at once. If it is not fully factorizable but factorizable we first apply (3.16) one or more times until we are left with an unfactorizable state and then we apply proposition 5 to the latter. We get another state which plays the role of $\Psi_{0,\bar{\nu}'}$. Proceeding further like this we establish (3.36) at the end. Analogously we consider $\Psi_{\bar{\varepsilon}',\bar{\nu}}$ from proposition 6 until we establish (3.36) for this case. □

The above enables us to show that the conditions of the theorem are *sufficient* for $d > d^-_{11}$. Indeed, in that case $\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} > 0$ for all states. What turns out to be important for the unitarity is that all counterterms are polynomials in d of lower degrees than $\mathcal{N}_{\bar{\varepsilon}\bar{\nu}}$ and all positivity requirements are determined by the terms $\mathcal{N}_{\bar{\varepsilon}\bar{\nu}}$. Unitarity at the reduction points will be considered in the next section.

4. Unitarity at the reduction points

4.1. The first reduction point

In this section we consider the unitarity of the irreps at the reducibility points d^-_{i1}. Unitarity is established by noting that there are no negative norm states and by factoring out the zero norm states which are a typical feature of the Verma modules V^Λ at the reducibility points. These zero norm states generate invariant submodules I_{i1} and are decoupled in the factor modules V^Λ/I_{i1} which realize the UIRs at the points $d = d^-_{i1}$.

In this subsection $d = d^-_{11}$, i.e. $x_{11} = 3$. We have the following:

Proposition 8. *Let $d = d^-_{11}$. There are no negative norm states. The zero norm states are described as follows. In the case $a_{\hat{k}} \neq 0$, $\hat{k} = 1, \dots, N$, the states of zero norm $\mathcal{F}^\Lambda_0$ from $\mathcal{F}^\Lambda$ are given by $\Psi_{\bar{\varepsilon}\bar{\nu}}$ with*

$$\varepsilon_{i\hat{\jmath}} = 0, 1, \qquad \nu_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = 1 \\ 0, 1 & \textit{otherwise}. \end{cases} \tag{4.1}$$

The number of such states is 2^{8N-4} *and the number of oddly generated states in the reduced irrep* $L^\Lambda \equiv \mathcal{F}^\Lambda \big/ \mathcal{F}_0^\Lambda$ *is* $15 \times 2^{8N-4}$.

In the cases $a_1 = \cdots = a_{\hat{k}} = 0, \hat{k} = 1, \ldots, N-1$*, in addition to those in (4.1) the following states have zero norm:*

$$\varepsilon_{i\hat{\jmath}} = 0, 1,$$

$$\nu_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = 2 \\ 0 & i = \hat{\jmath} = 1 \\ 0, 1 & \textit{otherwise} \end{cases} \tag{4.2.1}$$

and

$$\nu_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = 3 \\ 0 & i = 1, \hat{\jmath} = 1, 2 \\ 0, 1 & \textit{otherwise} \end{cases} \tag{4.2.2}$$

$\ldots$

$$\nu_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = \hat{k} + 1 \\ 0 & i = 1, \hat{\jmath} = 1, \ldots, \hat{k} \\ 0, 1 & \textit{otherwise.} \end{cases} \tag{4.2.$\tilde{k}$}$$

The number of states in (4.2.1), (4.2.2), $\ldots$*, (4.2.*$\tilde{k}$*) is* $2^{8N-5}, 2^{8N-6}, \ldots, 2^{8N-4-\tilde{k}}$*, resp., the overall number of states in (4.2) is* $2^{8N-4-\tilde{k}}(2^{\tilde{k}} - 1)$*, the number of states in the reduced* L^Λ *- factoring out both (4.1) and (4.2) - is* $2^{8N-4-\tilde{k}}(2^{4+\tilde{k}} - 2^{\tilde{k}+1} + 1)$.

In the case $r_1 = 0$ *(R-symmetry scalars) in addition to those in (4.1) and (4.2) for* $\hat{k} = N - 1$*, the following states have zero norm:*

$$\varepsilon_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = 1 \\ 0 & i = 1, \hat{\jmath} > 1 \\ 0, 1 & \textit{otherwise} \end{cases}$$

$$\nu_{i\hat{\jmath}} = \begin{cases} 0 & \hat{\jmath} = 1 \\ 0 & i = 1 \\ 0, 1 & \textit{otherwise} \end{cases} \tag{4.3.1}$$

and

$$\varepsilon_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = 2 \\ 0 & i = 1, \hat{\jmath} > 2 \\ 0, 1 & \textit{otherwise} \end{cases}$$

$$\nu_{i\hat{\jmath}} = \begin{cases} 0 & \hat{\jmath} = 2 \\ 0 & i = 1 \\ 0, 1 & \textit{otherwise} \end{cases} \tag{4.3.2}$$

$\ldots$

$$\varepsilon_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = N - 1 \\ 0 & i = 1, \hat{\jmath} = N \\ 0, 1 & \textit{otherwise} \end{cases}$$

$$\nu_{i\hat{\jmath}} = \begin{cases} 0 & \hat{\jmath} = N - 1 \\ 0 & i = 1 \\ 0, 1 & \textit{otherwise} \end{cases} \tag{4.3.$N-1$}$$

$$\varepsilon_{i\hat{\jmath}} = \begin{cases} 1 & \hat{\jmath} = N \\ 0, 1 & \textit{otherwise} \end{cases}$$

$$\nu_{i\hat{\jmath}} = \begin{cases} 0 & \hat{\jmath} = N \\ 0 & i = 1 \\ 0, 1 & \textit{otherwise.} \end{cases} \tag{4.3.N}$$

The number of states in (4.3.1), (4.3.2), ..., (4.3.N − 1), (4.3.N) is 2^{6N-6}, $2^{6N-5}, \ldots, 2^{7N-8}, 2^{7N-7}$, *resp., the overall number of states in (4.3) is* $2^{6N-6}(2^N - 1)$, *the number of states in the reduced* L^Λ*—factoring out (4.1), (4.2) (for* $\tilde{k} = N - 1$*) and (4.3)—is* $2^{6N-6}(2^{2N+6} - (2^{N+3} + 1)(2^N - 1))$.

Proof. There are no negative norm states if $d > d_{11}^-$ and thus there are no such states for $d = d_{11}^-$ by continuity. For the zero norm states we start with the case $a_{\hat{k}} \neq 0, \hat{k} = 1, \ldots, N$. Inspecting formula (3.21) we see that the fully factorized states of zero norm have the form (4.1). Indeed, the only factor in $\mathcal{N}_{\tilde{\varepsilon}\tilde{\nu}}$ that can be zero is $(x_{11} + \tilde{\nu}_{11}) = (3 + \tilde{\nu}_{11})$ (hence $\nu_{11} = 1$), which happens if $\tilde{\nu}_{11} = -3$ which happens if $\nu_{i1} = 1$, $i = 2, 3, 4$. (In general, $(x_{i\hat{\jmath}} + \tilde{\nu}_{i\hat{\jmath}}) \geqslant (3 + \tilde{\nu}_{i\hat{\jmath}}) \geqslant (i + \hat{\jmath} - 2)$.) Further, the problem is reduced to unfactorizable states. The main term of the norm squared is given again by $\mathcal{N}_{\tilde{\varepsilon}\tilde{\nu}}$ which is zero. For further use we note more explicitly that for the states from (4.1) we have

$$\mathcal{N}_{\tilde{\varepsilon}\tilde{\nu}} \sim (x_{11} - 3)(x_{21} - 2)(x_{31} - 1)x_{41}. \tag{4.4}$$

Now we shall show that the counterterms are also zero. For this it is enough to show that $\nu_{i1}^{m,n} = 1, i = 1, 2, 3, 4$ in all auxiliary states that happen in the counterterms. Consider first states starting with $X_{i\hat{\jmath}}^+$ for which the norm is given in proposition 6. The only way $\nu_{i1}^{m,n}$ could differ from ν_{i1} is if one of the pairs in (3.22) is of the form $(k_m, 1)$, more precisely, that could be only one of the pairs $(k_m, \hat{k}_{m,q(m)}) = (k_m, 1)$. But then according to (3.23) for any possible m we should have $\nu_{i,1} = 1$ and $\nu_{k_m,1} = 0$ which does not hold. Thus, all counterterms are also zero. Consider next states starting with $Y_{i\hat{\jmath}}^+$ for which the norm is given in proposition 7. Here only the counterterms in (3.32e, f) can possibly be non-zero. For the counterterm in (3.32e) the only way $\nu'^{m,n}_{i1}$ could differ from ν_{i1} is if one of the pairs in (3.28) is of the form $(k_m, 1)$, more precisely, that could be only one of the pairs $(k_m, \tilde{\ell}_{m,q'(m)}) = (k_m, 1)$. But then according to (3.29) for any possible m we should have $\nu_{i,1} = 1$ and $\nu_{k_m,1} = 0$ which does not hold. For the counterterm in (3.32f) the considerations are simpler since it is immediately seen from (3.30) that there is no pair that can affect $\nu_{i,1}$ since all $\tilde{\ell}_{m,n} > \hat{\jmath} \geqslant 1$, and if we consider $\hat{\jmath} = 1$ then our state does not fulfil the condition in (3.31) $\nu_{\ell_m,1} = 0$. Thus, all possible counterterms are zero and thus all states in (4.1) have zero norm. We continue with the cases $a_{\hat{k}} \neq 0$, $\hat{k} = 1, \ldots, N - 1$. Then $x_{1,\hat{k}+1} = \cdots = x_{12} = x_{11} = 3$. Under the hypothesis in (4.2.1) we have $\tilde{\nu}_{12} = -3$, hence $x_{12} + \tilde{\nu}_{12} = 0$ and the corresponding states have zero norm—the argument for unfactorizable states goes analogously to above. The same reasoning goes for all other cases in (4.2). For further use we note more explicitly that for the states from (4.2.$\hat{\ell}$), $\hat{\ell} = 1, 2, \ldots, \hat{k}$, we have

$$\mathcal{N}_{\tilde{\varepsilon}\tilde{\nu}} \sim (x_{1,\hat{\ell}+1} - 3)(x_{2,\hat{\ell}+1} - 2)(x_{3,\hat{\ell}+1} - 1)x_{4,\hat{\ell}+1}. \tag{4.5}$$

We continue with the case $r_1 = 0$. Then $y_{1,N} = \cdots = y_{11} = x_{11} = 3$. Under the hypothesis in (4.3.1) we have $\tilde{\varepsilon}_{11} = -3$, hence $y_{11} + \tilde{\nu}_{11} = 0$ and the corresponding states have zero norm—the argument for unfactorizable states goes analogously to above. The same reasoning goes for all other cases in (4.3). For further use we note more explicitly that for the states from (4.3.$\hat{\ell}$), $\hat{\ell} = 1, 2, \ldots, N$, we have

$$\mathcal{N}_{\tilde{\varepsilon}\tilde{\nu}} \sim (y_{1,\hat{\ell}} - 3)(y_{2,\hat{\ell}} - 2)(y_{3,\hat{\ell}} - 1)y_{4,\hat{\ell}}. \tag{4.6}$$

The counting of states is straightforward. □

4.2. The other reduction points

We first consider the case (3.8*b*) of the theorem: $d = d_{21}^-$ and $n_1 = 0$, i.e. $x_{11} = x_{21} = 2$. We have the following:

Proposition 9. *Let $d = d_{21}^-$ and $n_1 = 0$. There are no negative norm states. All states of zero norm which are described in proposition 8 have zero norm also under the present hypothesis. There are further states of zero norm which are described as follows. In the case $a_{\hat{k}} \neq 0$, $\hat{k} = 1, \ldots, N$, the additional states of zero norm are given by $\Psi_{\bar{\varepsilon}\bar{\nu}}$ with:*

$$\varepsilon_{i\hat{\jmath}} = 0, 1 \qquad \nu_{11} + \nu_{21} + \nu_{31} + \nu_{41} = 3 \qquad \nu_{i\hat{\jmath}} = 0, 1 \qquad \hat{\jmath} \neq 1. \tag{4.7}$$

The number of states in (4.7) is 2^{8N-2}, and the number of states in the reduced L^Λ—factoring out both (4.1) and (4.7)—is $11 \times 2^{8N-4}$.

In the case $a_1 = 0$ in addition to those in (4.7) the following states have zero norm for $N = 1$:

$$\varepsilon_{i1} = 1 \qquad \nu_{11} = 0 \qquad \nu_{21} + \nu_{31} + \nu_{41} = 1 \tag{4.8a}$$

and

$$\varepsilon_{11} + \varepsilon_{21} + \varepsilon_{31} + \varepsilon_{41} = 3 \qquad \nu_{i1} = 0. \tag{4.8b}$$

The number of states in (4.8a), (4.8b), is 3, 4, resp. the overall number of zero states—including (4.1), (4.3), (4.7), and (4.8)—is 88, and thus the number of states of the reduced L^Λ is 168.

Proof. We first have to show that the states of zero norm from proposition 8 have zero norm also here. With this we shall also establish that there are no negative norm states since those states are the only suspects for this. For the cases described by (4.1) this follows by inspecting (4.4) which is zero also here. In the cases described by (4.2$\hat{\ell}$), (4.3$\hat{\ell}$) this follows by inspecting (4.5), (4.6), which are zero also here. Further, the proof is as of proposition 8. In particular, for the states from (4.7) we have

$$\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} \sim \big(x_{i_1,1} - 2\big)\big(x_{i_2,1} - 1\big)x_{i_3,1} \tag{4.9}$$

where i_j are from the set 1, 2, 3, 4, and thus at least one of them is equal to 1 or 2, hence the RHS of (4.9) is zero. Analogously, for the states from (4.8*a*) holds (4.6) for $\hat{\ell} = 1$, hence $\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} = 0$. For the states from (4.8*b*) holds:

$$\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} \sim \big(y_{i_1,1} - 2\big)\big(y_{i_2,1} - 1\big)y_{i_3,1} \tag{4.10}$$

which is zero as (4.9) since $y_{i1} = x_{i1}$ for $a_1 = 0$. □

Next we consider the case (3.8*c*) of the theorem: $d = d_{31}^-$ and $n_1 = n_2 = 0$, i.e. $x_{11} = x_{21} = x_{31} = 1$. We have the following:

Proposition 10. *Let $d = d_{31}^-$ and $n_1 = n_2 = 0$. There are no negative norm states. All states of zero norm which are described in propositions 8 and 9 have zero norm also under the present hypothesis. There are further states of zero norm which are described as follows. In the case $a_{\hat{k}} \neq 0$, $\hat{k} = 1, \ldots, N$, the additional states of zero norm are given by $\Psi_{\bar{\varepsilon}\bar{\nu}}$ with*

$$\varepsilon_{i\hat{\jmath}} = 0, 1 \qquad \nu_{11} + \nu_{21} + \nu_{31} + \nu_{41} = 2. \tag{4.11}$$

The number of states in (4.11) is $3 \times 2^{8N-3}$, and the number of states in the reduced L^Λ—factoring out (4.1), (4.7) and (4.11)—is $5 \times 2^{8N-4}$.

In the case $a_1 = 0$ in addition to those in (4.11) the following states have zero norm for $N = 1$:

$$\varepsilon_{i1} = 1 \qquad \nu_{i1} = \delta_{i1} \qquad \text{and} \qquad \varepsilon_{11} + \varepsilon_{21} + \varepsilon_{31} + \varepsilon_{41} = 3 \qquad \nu_{11} + \nu_{21} + \nu_{31} + \nu_{41} = 1 \tag{4.12a}$$

$$\varepsilon_{11} = 1 \Rightarrow \nu_{11} = 0 \qquad \varepsilon_{11} = 0 \Rightarrow \nu_{21} = 0 \tag{4.12b}$$

and

$$\varepsilon_{11} + \varepsilon_{21} + \varepsilon_{31} + \varepsilon_{41} = 2 \qquad \nu_{i1} = 0. \tag{4.12c}$$

The number of states in (4.12a), (4.12b), (4.12c), is 1, 12, 6, resp., the overall number of zero states—including (4.1), (4.3), (4.7), (4.8), (4.11), (4.12)—is 203, and thus the number of states of the reduced L^Λ is 53.

Proof. We first have to show that the states of zero norm from propositions 8 and 9 have zero norm also here (establishing also the lack of negative norm states). For the cases described by (4.1), (4.2$\hat{\ell}$), (4.3$\hat{\ell}$), (4.7) (4.8) this follows by inspecting (4.4), (4.5), (4.6), (4.9), (4.10), which are zero also here. Further, the proof is as of propositions 8 and 9. In particular, for the states from (4.11) we have

$$\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} \sim \left(x_{i_1,1} - 1\right) x_{i_2,1} \tag{4.13}$$

where i_j are from the set 1, 2, 3, 4, and thus at least one of them is equal to 1 or 2 or 3, hence the rhs of (4.13) is zero. For the states from (4.12a), resp. (4.12b), hold (4.6) for $\hat{\ell} = 1$, resp. (4.10), hence $\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} = 0$. For the states from (4.12c) we have

$$\mathcal{N}_{\bar{\varepsilon}\bar{\nu}} \sim \left(y_{i_1,1} - 1\right) y_{i_2,1} \tag{4.14}$$

which is zero as (4.13) since $y_{i1} = x_{i1}$ for $a_1 = 0$. □

Finally we consider the case (3.8d) of the theorem: $d = d_{41}^-$ and $n_1 = n_2 = n_3 = 0$, i.e. $x_{11} = x_{21} = x_{31} = x_{41} = 0$. We have the following:

Proposition 11. *Let $d = d_{41}^-$ and $n_1 = n_2 = n_3 = 0$. There are no negative norm states. All states of zero norm which are described in propositions 8, 9 and 10 have zero norm also under the present hypothesis. There are further states of zero norm which are described as follows. In the case $a_{\hat{k}} \neq 0$, $\hat{k} = 1, \ldots, N$, the additional states of zero norm are given by $\Psi_{\bar{\varepsilon}\bar{\nu}}$ with*

$$\varepsilon_{i\hat{j}} = 0, 1 \qquad \nu_{11} + \nu_{21} + \nu_{31} + \nu_{41} = 1 \qquad \nu_{i\hat{j}} = 0, 1 \qquad \hat{j} \neq 1. \tag{4.15}$$

The number of states in (4.15) is 2^{8N-2}, and the number of states in the reduced L^Λ—factoring out (4.1), (4.7), (4.11) and (4.15)—is 2^{8N-4}.

In the case $a_1 = 0$ in addition to those in (4.15) the following states have zero norm for $N = 1$:

$$\varepsilon_{11} + \varepsilon_{21} + \varepsilon_{31} + \varepsilon_{41} = 1 \qquad \nu_{i1} = 0. \tag{4.16}$$

The number of states in (4.16) is 4, the overall number of zero states—including (4.1), (4.3), (4.7), (4.8), (4.11), (4.12), (4.15), (4.16)—is $2^8 - 1$ and thus the number of states of the reduced L^Λ is 1, i.e. this is the trivial representation.

Proof. We first have to show that the states of zero norm from propositions 8, 9 and 10 have zero norm also here (establishing also the lack of negative norm states). This is clear since in all cases the factor $\mathcal{N}_{\bar{\varepsilon}\bar{\nu}}$ contains as multiplicative factor some x_{i1} and hence is zero. The same holds for the states from (4.15). For $N = 1$ and $a_1 \neq 0$ there are 16 states which are of the form $\Psi_{\bar{\varepsilon},0}$. For $a_1 = 0$ all these, beside the vacuum state, are of zero norm since the factor $\mathcal{N}_{\bar{\varepsilon},0}$ contains as multiplicative factor some $y_{i1} = x_{y1} = 0$. For the counting of states we have to note that the 16 states in (4.8a) and (4.12a, b) are contained also in (4.15) if $a_1 = 0$. □

5. Outlook

In the previous subsection we gave the counting of states in the cases $a_{\hat{k}} \neq 0, \hat{k} = 1, \ldots, N$, only for $N = 1$. That would have taken many more pages due to the complicated combinatorics for $N > 1$ when $a_{\hat{k}} = 0$, and is left to a subsequent paper.

We also plan to construct the positive energy UIRs for $D = 3, 5$ conformal supersymmetry taking up the corresponding conjectures of Minwalla [43]. Other interesting objects are the conformal superalgebras for $D > 6$ recently introduced in [50].

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